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Characteristics of Nonlinear Positive Functionals and Their Applications

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1. INTRODUCTION

In 1966–1967, Beckenbach [3, 4] extended, among other things, the Hölder and Minkowski inequalities by a method of the differentiability of multivariable real functions. In 1979, Wang [21] established the *A–G* (arithmetic-geometric), Hölder, Minkowski, and Beckenbach inequalities by means of the convexity of multivariable real functions. However, their results dealt with only the discrete case of the inequalities. In the development of the theory of inequalities, the discrete and continuous cases of the inequalities have always been examined hand in hand (see Beckenbach and Bellman [5], Hardy, Littlewood, and Pólya [11], Iwamoto and Wang [13], and Mitrinović [16]), This examination has concerned not only the forms of the inequalities but also the methods used to establish them (e.g., see Iwamoto and Wang [13]). For this reason, we investigate here the continuous counterparts of the works of Beckenbach and Wang mentioned above. Indeed, the set of classical inequalities for integrals has been established by various other methods (e.g., see [5, 11, 13, 16]) as well. It is evident that the forms of inequalities for integrals can be regarded as nonlinear functionals (see [7, 14, 19]). On the other hand, in the study of nonlinear operators (and/or functionals), the differentiability of operators has been intensively and successfuly used to provide approximate solutions for nonlinear functional equations (e.g., see [7, 14, 19]), while their convexity has been used in a somewhat restricted manner (see Collatz [7, 334–341]). Although the differentiability and convexity of nonlinear functionals have attracted considerable attention of many investigators (see [7, 9, 14, 15, 17–19]), it seems to have escaped notice that they can be employed to establish the

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continuous versions of the discrete inequalities considered by Beckenbach [3, 4] and Wang [21]. This is the motivation of this paper.

To this end, in Section 2, we shall summarize notations and definitions that will be used. In Section 3, we shall present several theorems concerning the characteristics of nonlinear real functionals. In the following sections we shall successively establish the continuous versions of the A–G, Hölder, Minkowski, and Beckenbach inequalities by means of the theorems provided in Section 2 and conclude with some remarks.

2. Notations and Definitions

Let us begin by displaying some notations and symbols that we shall need:

$R =$ the field of real numbers.

$R^+ = \{(x \in R \mid x > 0\}$.

$R^n = \{(x_1, ..., x_n) \mid x_j \in R, j = 1, ..., n\}$.

$\Omega =$ an open convex subset of R^n.

$X = \{f \mid f: \Omega \to R^+\} =$ the set of integrable functions on Ω which is a Banach space by using the usual maximum norm (or sup norm, e.g., see [8, 25]).

$U =$ an open convex subset of X.

$\mathscr{F} = \{F \mid F: U \to R^+\} =$ the set of functionals on U.

$L_p = L_p(\Omega, S, \mu)$, $p > 0 =$ the space of all pth power positive integrable functions over a given finite measure space (Ω, S, μ).

For $f \in L_p$, we write

$$\|f\|_p = \left\{\int f^p \, d\mu\right\}^{1/p}.$$

Here and in what follows $\int$ and $d\mu$ are used to indicate $\int_\Omega$ and $d\mu(x) = dx_1 \cdots dx_n$, respectively, whenever confusion is unlikely to occur.

$$\langle f, g\rangle = \int fg \, d\mu, \qquad f \in L_p, \quad g \in L_{p/(p-1)}, \quad \text{or} \quad f, g \in X,$$

$$\phi(\lambda) = F(f + \lambda h), \qquad F \in \mathscr{F}, \quad f, \quad h \in X, \quad \lambda \in R,$$

$$\phi'(0) = \frac{d}{d\lambda} F(f + \lambda h)\,|_{\lambda=0} = \langle \partial_f F, h\rangle = \langle \partial F, h\rangle,$$

$$\phi''(0) = \frac{d^2}{d\lambda^2} F(f + \lambda h)\,|_{\lambda=0} = \langle \partial_f^2 F, h^2\rangle = \langle \partial^2 F, h^2\rangle = Q,$$

where $\phi'(0)$ and $\phi''(0)$ are said to be the first and second Gateaux differentials (e.g., see [19, p. 35]), respectively.

$$G(f) = \exp\left\langle \omega \Big/ \int \omega \, d\mu, \log f \right\rangle, \qquad \omega, \ \ f \in U.$$

DEFINITION 1. The functional $A: \mathscr{F} \to R$ is affine if for every $f \in \mathscr{F}$, $A(f) = L(f) + b$, where $L: \mathscr{F} \to R$ is linear and b is a constant in R.

DEFINITION 2. A functional $F: U \to R^+$ has support at $g \in U$ if there is an affine functional $A: \mathscr{F} \to R$ such that $A(g) = F(g)$ and $A(f) \leqslant F(f)$ (or $A(f) \geqslant F(f)$) for every $f \in U$.

We also refer to Collatz [7], Kantorovich and Akilov [14], and Vainberg [19] for the properties of the Gateaux differentials in particular and definitions of nonlinear functionals (or operators) in general used throughout the paper (unless specified otherwise).

For the classical inequalities cited here without mentioning any source, one should refer to Beckenbach and Bellman [5], Hardy *et al.* [11], or Mitrinović [16] for details.

3. THEOREMS CONCERNING NONLINEAR POSITIVE FUNCTIONALS

Corresponding to the definition, existance, and properties of the Gateaux differential of a nonlinear operator given in Vainberg [19, pp. 35–53], we can spell out the same for the Gateaux differential $\langle \partial_f F, g \rangle$, f, $g \in X$, of a nonlinear positive functional F of $\mathscr{F}$. However, we omit the nearly repetitive detail and refer to Vainberg [19] for it.

We now adopt the idea of Vainberg [19] and create several theorems (see also Courant and John [8], Ekeland and Teman [9], Kantorovich and Akilov [14], Lee [15], Robert and Varberg [17] and Rockafellar [18] for comparisons) concerning nonlinear positive functionals as follows for our purpose.

THEOREM 1. *Let F have continuous second Gateaux differential on U. Then a point $\xi \in U$ at which $\partial F(\xi)$ vanishes is a maximum point of F if $\partial^2 F(\xi) < 0$ throughout U, and a minimum point of F if throughout U $\partial^2 F(\xi) > 0$.*

Remark. $\partial F(\xi) = 0$ iff $\langle \partial F(\xi), h \rangle = 0$ for any $h \in \mathscr{F}$; $\partial^2 F(\xi) > 0$ iff $< \partial^2 F(\xi), h^2 \rangle > 0$ for any $h \in \mathscr{F}$; etc.

THEOREM 2. *Let F have continuous second Gateaux differential on U. Then F is convex (or concave) iff the quadratic form Q is nonnegative (or* *nonpositive) for all* $f \in U$, $h \in \mathscr{F}$. *Moreover, if* Q *is positive (or negative) on* U, *then* F *is strictly convex (or concave).*

THEOREM 3. *The nonlinear positive functional* F *is convex (or concave) on* U *iff* F *has support* $A(f)$ *at each point* $\xi \in U$, $A(\xi) = F(\xi)$ *and* $A(f) \leqslant F(f)$ *(or* $A(f) \geqslant F(f)$*) for every* $f \in U$.

In view of the Taylor formula of F

$$F(f) = F(\xi) + \langle \partial F(\xi), f - \xi \rangle + \tfrac{1}{2} \langle \partial^2(\theta\xi), (f - \xi)^2 \rangle, \qquad 0 < \theta < 1$$

proofs of Theorems 1–3 can be easily given by transplantation from those for ordinary functions (e.g., see Courant and John [8, p. 243]; Robert and Varberg [17, pp. 103, 108]) and a routine repetition is thus omitted.

4. A–G INEQUALITY

We state and prove A–G inequality as follows:

THEOREM 4. *If the functional* $\langle \omega / \int \omega \, d\mu, f \rangle$ *is finite, then*

$$G(f) \leqslant \left\langle \omega \Big/ \int \omega \, d\mu, f \right\rangle, \qquad \omega, f \in U \tag{1}$$

with equality iff $f = k$, *where* $k \in R^+$.

To prove Theorem 4, we consider the functional

$$G(f) = \exp \left\langle \omega \Big/ \int \omega \, d\mu, \log f \right\rangle. \tag{2}$$

Since

$$\partial G = G \int \omega f^{-1} \, d\mu \Big/ \int \omega \, d\mu,$$

and

$$\partial^2 G = \frac{G}{(\int \omega \, d\mu)^2} \left[\left(\int \omega f^{-1} \, d\mu \right)^2 - \int \omega \, d\mu \int \omega f^{-2} \, d\mu \right],$$

$$Q = \frac{G}{(\int \omega \, d\mu)^2} \left[\left(\int \omega f^{-1} h \, d\mu \right)^2 - \int \omega \, d\mu \int \omega f^{-2} h^2 \, d\mu \right]. \tag{3}$$

By a use of the Pythagorean equality (e.g., see Wang [24, p. 98]) on (3), we have $Q < 0$. Thus G is concave on U and the support of G at $f = k$ is

$$A(f) = \partial G(k)(f - k) + G(k) = \left\langle \omega \Big/ \int \omega \, d\mu, f - k \right\rangle + k = \left\langle \omega \Big/ \int \omega \, d\mu, f \right\rangle. \quad (4)$$

Hence inequality (1) is established by a use of Theorem 3 on (2) and (4). Also, the sign of equality in (1) holds iff $f = k$.

Remark. A simple manipulation of the form given in [...] of (3) yields the identity

$$\int \omega \, d\mu \langle \omega f^{-2}, h^2 \rangle - \langle \omega f^{-1}, h \rangle^2$$
$$= \int \left(\omega^{1/2} f^{-1} h \sqrt{\int \omega \, d\mu} - \left\langle \omega^{1/2} \Big/ \sqrt{\int \omega \, d\mu}, \omega^{1/2} f^{-1} h \right\rangle \omega^{1/2} \right)^2 d\mu \quad (5)$$

which is indeed a continuous version of the discrete Lagrange identity (see, e.g., Beckenbach and Bellman [5, p. 3] or Wang [21, p. 359]).

5. Hölder Inequality

We state and prove Hölder inequality as follows:

THEOREM 5. *If $p > 1$, then*

$$\left\langle f, \frac{g}{\|g\|_q} \right\rangle \leqslant \|f\|_p, \qquad f \in U \cap L_p, \quad g \in U \cap L_q, \quad (6)$$

where g is given and $q = p/(p-1)$. The inequality sign in (6) *is reversed for $0 < p < 1$. In either case the sign of equality holds iff $f^p = c^p g^q$, where $c \in R^+$.*

To prove Theorem 5, we consider the functional

$$F(f) = \|f\|_p.$$

Since $\partial F = \|f\|_p^{1-p} \int f^{p-1} \, d\mu$, and

$$\partial^2 F = (p-1) \|f\|_p^{1-2p} \left[\|f\|_p^p \int f^{p-2} \, d\mu - \left(\int f^{p-1} \, d\mu \right)^2 \right],$$
$$Q = (p-1) \|f\|_p^{1-2p} [\cdots],$$

where by a Lagrange identity similar to that of (5)

$$|\cdots| = \int f^p \, d\mu \int f^{p-2} h^2 \, d\mu - \left(\int f^{p-1} h \, d\mu \right)^2 > 0.$$

The support of F at $f = cg^{q/p}$ is

$$\begin{aligned} A(f) &= \langle \partial F(cg^{q/p}), f - cg^{q/p} \rangle + F(cg^{q/p}) \\ &= \left\langle \frac{g}{\| g \|_q}, f - cg^{q/p} \right\rangle + c \, \| g \|_q^{q/p} \\ &= \left\langle f, \frac{g}{\| q \|_q} \right\rangle. \end{aligned}$$

From the above, it follows that if $p > 1$, $Q > 0$ and if $p < 1$, $Q < 0$. Hence the theorem follows from Theorems 2 and 3, i.e., $A(f) \leqslant F(f)$ for $p > 1$ and $A(f) \geqslant F(f)$ for $p < 1$; in either case, $A(f^*) = F(f^*)$, where $f^* = cg^{q/p}$.

6. Minkowski Inequalities

We state and prove three Minkowski inequalities as follows (see Beckenbach [1] and Wang [21] for a comparison):

THEOREM 6. *For $\omega, f, g \in U$, we have*

$$G(f+g) \geqslant G(f) + G(g)$$

with equality iff $f = k_1$ and $g = k_1$ and $g = k_2$, where $k_1, k_2 \in R^+$.

THEOREM 7. *If $p > 1$, then*

$$\| f + g \|_p \leqslant \| f \|_p + \| g \|_p, \qquad f, g \in U \cap L_p. \tag{7}$$

The inequality sign in (7) is reversed for $0 < p < 1$. In either case, the sign of equality holds iff $f = kg$, where $k \in R^+$.

THEOREM 8. *Let $M_t(f)$ be the weighted mean of order t of the function f on Ω with weight $w \in U$ defined as follows:*

$$M_t(f) = \left\langle \omega \Big/ \int \omega \, d\mu, \quad f^t \right\rangle^{1/t}, \qquad t \neq 0, \quad t \in R,$$

$$M_0(f) = G(f).$$

Then it satisfies Minkowski inequality

$$M_t(f+g) \leqslant M_t(f) + M_t(g), \qquad f, g \in U \tag{8}$$

for $t > 1$. *The inequality sign is reversed in* (8) *for* $t \leqslant 1$. *In either case the sign of equality holds iff* $f = kg$, *where* $k \in R^+$.

Proofs of Theorems 6 and 7 are identical with those of Theorems 4 and 5 gives in Wang [21, p. 359], respectively. We omit the detail.

To prove Theorem 8, we consider the functional

$$F(f) = M_t = M_t(f).$$

Since $\partial F = M_t^{1-t} \int \omega f^{t-1}\, d\mu / \int \omega\, d\mu$, and

$$\partial^2 F = (t-1) M_t^{1-2t} \left[\int \omega f^t d\mu \int \omega f^{t-2}\, d\mu - \left(\int \omega f^{t-1}\, d\mu \right)^2 \right] \Big/ \left(\int \omega\, d\mu \right)^2,$$

$$Q = (t-1) M_t^{1-2t} [\cdots] \Big/ \left(\int \omega\, d\mu \right)^2, \tag{9}$$

where by a Lagrange identity similar to that of (5)

$$[\cdots] = \int \omega f^t\, d\mu \int \omega f^{t-2} h^2\, d\mu - \left(\int \omega f^{-1}\, h\, d\mu \right)^2 > 0.$$

From (9), it follows that $Q > 0$ if $t > 1$ and $Q \leqslant 0$ if $t \leqslant 1$. Hence, by Theorem 2, $M_t(f)$ is convex in f for $t > 1$ and concave in f for $t \leqslant 1$. Combining the above with $M_t(rf) = rM_t(f)$ for all $r \in R^+$, the conclusion of the theorem is now clear.

7. Beckenbach Inequality

One of the most interesting generalizations of the discrete Hölder inequality was proved by E. F. Beckenbach as Theorem 1 given on page 24 of [3]. This generalization is now known as Beckenbach inequality (see Mitrinović [16] and Wang [20, 21] also). We cite it in a slightly modified manner as

THEOREM 9. *Let* $\alpha, \beta, A, B, k_j \in R^+$, $m+1 \leqslant j \leqslant n$, $0 \leqslant m < n$ be given and let p and q satisfy $q = p/(p-1)$. Then for $p > 1$, the Beckenbach inequality

$$\frac{(\alpha A + \beta \sum_{j=m+1}^n x_j^p)^{1/p}}{\alpha B + \beta \sum_{j=m+1}^n k_j x_j} \geqslant \frac{(\alpha A + \beta \sum_{j=m+1}^n d_j^p)^{1/p}}{\alpha B + \beta \sum_{j=m+1}^n k_j d_j}, \tag{10}$$

where

$$d_j = (Ak_j/B)^{q/p}, \qquad m+1 \leqslant j \leqslant n,$$

holds for all $x_{m+1},..., x_n \in R^+$. The inequality sign in (10) is reversed for $0 < p < 1$. In either case, the sign of equality holds iff

$$x_j = d_j, \qquad m+1 \leqslant j \leqslant n.$$

Remark. Setting $\alpha = \beta = 1$, $A = \sum_{j=1}^{m} c_j^p$, $B = \sum_{j=1}^{m} k_j c_j$, where $c_1,..., c_m \in R^+$ are given, and $d_j = \bar{c}_j$ $(m+1 \leqslant j \leqslant n)$ in Theorem 9, we obtain Theorem 1 of Beckenbach [3, p. 24].

Transliterating from Theorem 9, we formulate the continuous version of Beckenbach inequality as

THEOREM 10. *Let α, β, A, $B \in R^+$ and $k \in U$ be given and let p and q satisfy $q = p/(p-1)$. Then for $p > 1$, the inequality*

$$F(f) \geqslant F(d), \tag{11}$$

where

$$F(f) = \frac{(\alpha A + \beta \|f\|_p^p)^{1/p}}{\alpha B + \beta < f, K\rangle} \tag{12}$$

and

$$d(x) = [Ak(x)/B]^{p/q}, \qquad x \in \Omega$$

holds for $f \in U \cap L_p$. The inequality sign in (11) *is reversed for* $0 < p < 1$. *In either case, the sign of equality holds iff $f = d$.*

We now give two proofs of Theorem 10. First, we consider the functional $F(f)$ given in (12). Simple manipulations reveal that

$$\partial F = \beta P(f)\, S(f)$$

and

$$\partial^2 F = \beta(p-1)\frac{(\alpha A + \beta \|f\|_p^p)^{(1/p)-2}}{\alpha B + \beta\langle f, k\rangle}\, T(f) - \frac{2\beta^2 \int k\, d\mu}{\alpha B + \beta\langle f, K\rangle}\, P(f)\, S(f),$$

where

$$P(f) = \frac{(\alpha A + \beta \|f\|_p^p)^{(1/p)-1}}{(\alpha B + \beta < f, k\rangle)^2},$$

$$S(f) = (\alpha B + \beta\langle f, k\rangle)\int f^{p-1}\,d\mu - (\alpha A + \beta\,\|f\|_p^p)\int k\,d\mu$$

$$= \left\langle f^{p-1} - Ak/B, \alpha B + \beta\int kf\,d\mu - \beta f\int k\,d\mu\right\rangle$$

and

$$T(f) = (\alpha A + \beta\,\|f\|_p^p)\int f^{p-2}\,d\mu - \beta\left(\int f^{p-1}\,d\mu\right)^2 \tag{13}$$

$$= \alpha A\int f^{p-2}\,d\mu + \beta\int\left(\|f\|_p^{p/2} f^{(p-2)/2} - \left\langle \frac{f^{p/2}}{\|f\|_p^{p-2}}, f^{(p-2)/2}\right\rangle f^{p/2}\right)^2 d\mu.$$

Since $P(f) > 0$, $\partial F(f) = 0$ iff $S(f) = 0$; i.e., $\partial F(d) = 0$,

$$\partial^2 F(d) = \beta(p-1)\frac{(\alpha A + \beta\,\|d\|_p^p)^{(1/p)-2}}{\alpha B + \beta\langle f, d\rangle}\,T(d) \qquad \text{and} \qquad T(d) > 0,$$

$\partial^2 F(d) > 0$ for $p > 1$ and $\partial^2 F(d) < 0$ for $p < 1$. Hence, by Theorem 1, $F(f)$ has a unique minimum value for $p > 1$ and a unique maximum value for $p < 1$ at $f = d$. The theorem is thus proved.

To prove Theorem 10 alternatively, we consider the functional

$$\hat{F}(f) = (\alpha A + \beta\,\|f\|_p^p)^{1/p}. \tag{14}$$

Since $\partial\hat{F} = \beta\hat{F}^{1-p}\int f^{p-1}\,d\mu$ and

$$\partial^2\hat{F} = \beta(p-1)\,\hat{F}^{1-2p}T(f),$$

where $T(f)$ is given in (13),

$$Q = \langle\partial^2\hat{F}, h^2\rangle = \beta(p-1)\,\hat{F}^{1-2p}\langle T, h^2\rangle. \tag{15}$$

From (15), it follows that $Q > 0$ if $p > 1$ and $Q < 0$ if $p < 1$. Moreover, the support of $\hat{F}$ at $f = d$ is

$$\begin{aligned}
A(f) &= \hat{F}(d) + \partial\hat{F}(d)(f-d)\\
&= \frac{\hat{F}(d)}{\hat{F}^p(d)}\,[\hat{F}^p(d) + \beta < d^{p-1}, f-d\rangle]\\
&= \frac{\hat{F}(d)}{\alpha A + \beta(A/B)\langle d, k\rangle}\left[\alpha A + \beta\frac{A}{B}\langle f, k\rangle\right]\\
&= \frac{\hat{F}(d)}{\alpha B + \beta\langle d, k\rangle}(\alpha B + \beta\langle f, k\rangle).
\end{aligned} \tag{16}$$

Applying Theorem 3, we obtain that $A(f) \geqslant \hat{F}(f)$ if $p > 1$ and $A(f) \leqslant \hat{F}(f)$ if $p < 1$. In either case, $A(d) = \hat{F}(d)$ from (16). The conclusion of the theorem is now clear.

8. Concluding Remarks

For the other theorems given in Beckenbach [3, 4], their continuous versions can be established likewise by means of Theorem 1 and details are thus omitted.

It is also understood that a semi-norm in convex (see, e.g., Yosida [26, p. 24]) but a convex functional needs not to be a semi-norm. For example, the convex functional $\hat{F}$ (given in (14)) is not a semi-norm.

All the conditions given above for the equality case of each related inequality are stated such as "$f = kg$." In fact, it can be replaced by "$f = kg$ almost everywhere" (e.g., see Halmos [10, p. 86] or Yosida [25, p. 15]). Finally, let us point out that in turn Theorem 9 follows immediately from Theorem 10 by letting $f(x) = \sum_{j=m+1}^{n} x_j \chi_{[j,j+1)}$, $k(x) = \sum_{j=m+1}^{n} k_j \chi_{[j,j+1)}$ (see Halmos [10, p. 84]), $d\mu = dx$, $x \in R$, and $\Omega = [m+1, n+1)$.

The interrelation between the discrete and continuous cases of inequalities and among inequalities themselves (e.g., see [5, 11, 16, 22, 23]) will be drawn a little closer by the above results.

References

1. E. F. Beckenbach, A class of mean-value function, *Amer. Math. Monthly* **57** (1950), 1–6.
2. E. F. Beckenbach, On the inequality of Kantorovich, *Amer. Math. Monthly* **11** (1964), 606–619.
3. E. F. Beckenbach, On Hölder's inequality, *J. Math. Anal. Appl.* **15** (1966), 21–29.
4. E. F. Beckenbach, A "Workshop" on Minkowski's inequality, *in* "Inequalities I" (O. Shisha, Ed.), pp. 37–55, Academic Press, New York, 1967.
5. E. F. Beckenbach and R. Bellman, "Inequalities," 2nd rev. ed., Springer-Verlag, Berlin, 1965.
6. R. Bellman, "Dynamic Programming," Princeton Univ. Press, Princeton, N.J., 1957.
7. L. Collatz, "Functional Analysis and Numerical Mathematics" (Translated by H. Oser), Academic Press, New York, 1966.
8. R. Courant and F. John, "Calculus and Analysis," Vol. I, Wiley–Interscience, New York, 1965.
9. I. Ekeland and R. Teman, "Convex Analysis and Variational Problems," North-Holland/American Elsevier, Amsterdam/New York, 1976.
10. P. R. Halmos, "Measure Theory," Van Nostrand, Princeton N.J., 1950.
11. G. H. Hardy, J. E. Littlewood, and G. Pólya, "Inequalities," 2nd ed., Cambridge Univ. Press, Cambridge, 1952.
12. S. Iwamoto, Dynamic programming approach to inequalities, *J. Math. Anal. Appl.* **58** (1977), 687–704.

13. S. IWAMOTO AND C.-L. WANG, Continuous dynamic programming approach to inequalities, submitted.
14. L. V. KANTOROVICH AND G. P. AKILOV, "Functional Analysis in Normed Spaces" (Translated by D. E. Brown), Macmillan Co., New York, 1965.
15. W. Q. LEE, "Functional Analysis," rev. ed., Science Pub. Co., 1966. [Chinese]
16. D. S. MITRINOVIĆ, "Analytic Inequalities," Springer-Verlag, Berlin, 1970.
17. A. W. ROBERTS AND D. A. VARBERG, "Convex Functions," Academic Press, New York, 1973.
18. R. T. ROCKAFELLAR, "Convex Analysis," Princeton Univ. Press, Princeton, N. J., 1970.
19. M. M. VAINBERG, "Variational Methods for the Study of Nonlinear Operators" (Translated by A. Feinstein), Holden–Day, San Francisco, 1964.
20. C. L. WANG, Functional equation approach to inequalities, *J. Math. Anal. Appl.*. **71** (1979), 423–430.
21. C.-L. WANG, Convexity and inequalities, *J. Math. Anal. Appl.* **72** (1979), 355–361.
22. C.-L. WANG, On development of inverses of the Cauchy and Hölder inequalities, *SIAM Rev.* **21** (4) (1979), 550–557.
23. C.-L. WANG, A survey on basic inequalities, Notes, *Canad. Math. Soc.* **12** (9) (1980), 8–12.
24. C.-L. WANG, Gramian expansions and their applications, *Utilitas Math.* **15** (1979), 97–111.
25. K. YOSIDA, "Functional Analysis," 2nd printing cor., Springer-Verlag, Berlin, 1966.